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Phase Transitions and Reflection Positivity for a Class of Quantum Lattice Systems

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We prove a form of reflection positivity in planes containing sites for a class of quantum lattice systems. As an application, a proof is given of a phase transition for the Fisher-stabilized Ising antiferromagnet in an external magnetic field with parallel and transverse components, both by the method of infrared bounds and by a suitable version of the Peierls argument. We also discuss the spherical model in an appendix.

KEY WORDS: Phase transition; reflection positivity; quantum lattice systems; antiferromagnets; infrared bounds; peierls arguments.

1. INTRODUCTION AND SUMMARY

In a pioneer paper,[1] Froehlich, Simon, and Spencer proved for the first time the existence of phase transitions for classical lattice systems with continuous symmetry. Their method was further generalized to include a class of quantum lattice systems by Dyson, Lieb, and Simon[2] and later abstracted and generalized to include proofs of phase transitions for both classical and quantum lattice systems by Froehlich, Israel, Lieb, and Simon.[4] In the latter, the property of reflection positivity (RP) was most clearly isolated as a central element of both the proofs employing the method of infrared bounds,[1,4,5] as well as those which involve generalized versions of the Peierls argument.[3–5]

In many applications, RP is required to be in planes between lattice sites.[1,2] In Ref. 3 and especially 5 applications were considered which require RP *in planes containing sites*. As remarked in Ref. 5, this seems to pose the unfortunate limitation that quantum systems are not allowed. In this paper we consider, however, a class of quantum lattice systems requiring RP in planes containing sites, which involve typically a transverse

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magnetic field. Although somewhat restricted, this class illustrates a method which might be of wider range of application.

The paper is organized as follows. In Section 2 we prove our main result, which is a form of RP in planes containing (or not) lattice sites for a class of models (Theorem 2.1). The method of proof may be roughly described as rendering the system "classical" by means of the Trotter product formula, together with a choice of convenient intermediate states (alternatively, path-space methods similar to Ref. 6 could have been used).

In Sections 3 and 4 we illustrate the results through a typical model, namely, the Fisher-stabilized Ising antiferromagnet (see Ref. 5) in an external magnetic field with both parallel and transverse components. In Section 3 we employ the method of infrared bounds. There the "classical version" of the model is also used for the purpose of proving some inequalities along the lines of Ref. 6 which are necessary for the proof. Motivation for inequalities of this type stems from the similar structure of the spherical model with external parallel field, which is discussed for completeness in Appendix A. In Section 4 we sketch the proof of a phase transition for the model without stabilization, using a version of the Peierls argument developed in Refs. 3 and 5.

2. REFLECTION POSITIVITY IN PLANES CONTAINING SITES: REAL QUANTUM SYSTEMS

The notation and terminology of this section follows with minor modifications the one adopted in Ref. 4. Let $\mathcal{Q}$ be a real algebra with unit which in our applications will be typically non-Abelian, and let $\hat{\mathcal{Q}}$ be an *Abelian* subalgebra of $\mathcal{Q}$. Given a linear functional on $\mathcal{Q} : A \to \langle A \rangle_0$ with $\langle \mathbb{1} \rangle_0 = 1$ and $H \in \mathcal{Q}$, we define

$$\langle A \rangle_H \equiv \langle A e^{-H} \rangle_0 / \langle e^{-H} \rangle_0 \tag{2.1}$$

We suppose that $\mathcal{Q}$ contains two subalgebras $\mathcal{Q}_+$ and $\mathcal{Q}_-$ and a real linear morphism θ on $\mathcal{Q}_+ \cup \mathcal{Q}_-$ (the smallest subalgebra of $\mathcal{Q}$ containing both $\mathcal{Q}_+$ and $\mathcal{Q}_-$) such that

(a) $\theta(\mathcal{Q}_+) = \mathcal{Q}_-$

(b) $\theta^2 = \mathbb{1}$

(c) $\langle \theta A \rangle_0 = \langle A \rangle_0 \qquad \forall A \in \mathcal{Q}$

(d) $\theta(\hat{\mathcal{Q}}_+) = \hat{\mathcal{Q}}_-$

where $\hat{\mathcal{Q}}_\pm \equiv \hat{\mathcal{Q}} \cap \mathcal{Q}_\pm$

Definition 2.1. A real linear functional $\langle \cdot \rangle$ on $\mathcal{A}$ is called $\hat{\mathcal{A}}$-reflection positive iff

$$\langle A\theta(A)\rangle \geqslant 0 \tag{2.2}$$

for all $A \in \hat{\mathcal{A}}_+$. $\langle \cdot \rangle$ is called $\hat{\mathcal{A}}$-generalized reflection positive iff

$$\langle A_1\theta(A_1) \cdots A_n\theta(A_n)\rangle \geqslant 0 \qquad \text{for all } A_1, \ldots, A_n \in \hat{\mathcal{A}}_+ \tag{2.3}$$

Remark 2.1. (1) It is important to notice that we do not assume $\mathcal{A}_+$ and $\mathcal{A}_-$ to commute with each other and this is the reason why we consider this restricted form of reflection positivity.

(2) Since $\hat{\mathcal{A}}$ is Abelian, $\langle \cdot \rangle$ is $\hat{\mathcal{A}}$-reflection positive iff $\langle \cdot \rangle$ is $\hat{\mathcal{A}}$-generalized reflection positive.

If follows from the above definitions[4] that if $-H = B + \theta B + \sum_{i=1}^{k} C_i\theta(C_i)$ with $B_1 C_i \in \hat{\mathcal{A}}_+$ then $\langle \cdot \rangle_H$ is $\hat{\mathcal{A}}$-reflection positive. The aim of the following discussion is to extend this result, allowing B to be certain operators in $\mathcal{A}_+$ rather than in $\hat{\mathcal{A}}_+$.

We will consider the case where $\mathcal{A}$ is the algebra of observables of a quantum system composed of three "parts," that is, its Hilbert space of states $\mathcal{H}$ is given by $\mathcal{H} = \mathcal{H}_- \otimes \mathcal{H}_0 \otimes \mathcal{H}_+$ where $\mathcal{H}_+$ and $\mathcal{H}_-$ are isomorphic, and $\mathcal{H}_\pm, \mathcal{H}_0$ are all finite dimensional. $\mathcal{A}$ is the algebra of all real operators on $\mathcal{H}$.

$\mathcal{A}_+$ is the algebra generated by all operators in $\mathcal{A}$ of the form $\mathbb{1} \otimes A \otimes B$ (under the decomposition $\mathcal{H}_- \otimes \mathcal{H}_0 \otimes \mathcal{H}_+$). Since $\mathcal{A}$ is the linear span of operators of the form $A \otimes B \otimes C$, θ is well defined by

$$\theta(A \otimes B \otimes C) = C \otimes B \otimes A \tag{2.4}$$

If $\langle A\rangle_0 = \mathrm{Tr}_{\mathcal{H}} A / \mathrm{Tr}_{\mathcal{H}} \mathbb{1}$, then properties (a), (b), and (c) listed above are trivially verified. Let $\hat{\mathcal{A}}_+$ be a commutative subalgebra of $\mathcal{A}_+$, $\hat{\mathcal{A}}_- = \theta\hat{\mathcal{A}}_+$ and $\hat{\mathcal{A}} = \hat{\mathcal{A}}_+ \cup \hat{\mathcal{A}}_-$. Then property (d) is also fulfilled.

In order to state the main result of this section we introduce further the subalgebra $\mathcal{B}_+$ as the set of elements in $\mathcal{A}_+$ of the form $\mathbb{1} \otimes \mathbb{1} \otimes A$, $\mathcal{B}_- = \theta\mathcal{B}_+$ and $\mathcal{B}_0$ as the set of operators in $\mathcal{A}$ of the form $\mathbb{1} \otimes A \otimes \mathbb{1}$. Then $\hat{\mathcal{B}}_0 = \mathcal{B}_0 \cap \hat{\mathcal{A}}$.

A bounded operator A on a Hilbert space $\mathcal{H}$ is called positivity preserving with respect to a basis $\{\varphi_n\}_{n\geqslant 1}$ in $\mathcal{H}$ iff $(\varphi_n, A\varphi_m) \geqslant 0$ for all $n, m \geqslant 1$.

Theorem 2.1. Let $B \in \mathcal{B}_+$, $B_0 \in \mathcal{B}_0$ and e^{tB_0} for all $t > 0$ be a positivity-preserving operator in $\mathcal{H}_0$ with respect to a basis $\{\varphi_n^0\}_{n\geqslant 1}$ which diagonalizes $\hat{B}_0$. If

$$-H = B + \theta B + B_0 + \sum_{i=1}^{N} C_i\theta C_i + D + \theta D \tag{2.5}$$

with $C_i, D \in \hat{\mathcal{A}}_+$, then $\langle \cdot \rangle_H$ is $\hat{\mathcal{A}}$-reflection positive.

Proof. Let $\{\Psi_{\underline{n}} = \varphi_{n_-} \otimes \varphi^0_{n_0} \otimes \varphi_{n_+}, \underline{n} \equiv (n_-, n_0, n_+)\}$ be a basis which diagonalizes $\hat{\mathscr{Q}}$. We first notice that if $V = B + \theta B + B_0$ then

$$\left(\Psi_{\underline{n}}, e^{tV}\Psi_{\underline{m}}\right) = b^t(n_-, m_-)b_0^t(n_0, m_0)b^t(n_+, m_+) \tag{2.6}$$

with $b^t(n,m) = (\varphi_n, e^{tB}\varphi_m)$ and $b_0^t(n,m) = (\varphi_n^0, e^{tB_0}\varphi_m^0)$. Moreover if $-H_0 = \sum C_i\theta C_i + D + \theta D$ then

$$\left(\Psi_{\underline{n}}, e^{-tH_0}\Psi_{\underline{m}}\right) = \delta_{\underline{n},\underline{m}}F^t(\underline{n}) \tag{2.7}$$

where the function $F_{\underline{n}}$ can be written in the form

$$F_{\underline{n}}(t) = \sum f_i^t(n_0, n_-)f_i^t(n_0, n_+) \tag{2.8}$$

We may further suppose the matrix elements $(\Psi_{\underline{n}}, A\Psi_{\underline{m}})$ to be real for all $A \in \mathscr{Q}$, since $\mathscr{Q}$ is an algebra of real operators.

Using Trotter product formula we have then

$$\begin{aligned}\langle A\theta(A)\rangle_H\langle e^{-H}\rangle &= \operatorname{Tr} A\theta(A)e^{-H}\\ &= \lim_{K\to\infty} \operatorname{Tr} A\theta(A)\left[e^{-H_0/K}e^{-V/k}\right]^k\end{aligned} \tag{2.9}$$

From (2.6), (2.7), and (2.8) we get

$$\begin{aligned}\operatorname{Tr} A\theta(A)\left[e^{-H_0/k}e^{-V/k}\right]^k &= \sum_{\underline{n}^1}\left\langle \Psi_{\underline{n}^1}, A\theta(A)\left[e^{-H_0/k}e^{-V/k}\right]^k\Psi_{\underline{n}^1}\right\rangle\\ &= \sum_{\underline{n}^1,\underline{n}^2,\ldots,\underline{n}^k} a\left(n_0^1, n_-^1\right)a\left(n_0^1, n_+^1\right)F^t\left(\underline{n}^1\right)b^t\left(n_-^1, n_-^2\right)\\ &\quad\times b_0^t\left(n_0^1, n_0^2\right)b^t\left(n_+^1, n_+^2\right)\cdots F^t\left(\underline{n}^k\right)b^t\left(n_-^k, n_-^1\right)\\ &\quad\times b_0^t\left(n_0^k, n_0^1\right)b^t\left(n_+^k, n_+^1\right)\\ &= \sum_{\underline{i}}\ \sum_{n_0^1, n_0^2, \ldots, n_0^k} b_0^t\left(n_0^1, n_0^2\right)\\ &\quad\cdots b_0^t\left(n_0^k, n_0^1\right)\left[g_{\underline{i}}\left(n_0^1, \ldots, n_0^k\right)\right]^2 \geqslant 0\end{aligned} \tag{2.10}$$

where the first summation sign $\sum_{\underline{i}}$ refers to the sums we get by using formula (2.8) k times and the $g_{\underline{i}}$ are functions of the type

$$\begin{aligned}\sum_{n^1,\ldots,n^K} & a\left(n_0^1, n^1\right)f_{i_1}^t\left(n_0^1, n^1\right)b^t(n^1, n^2)f_{i_2}\left(n_0^2, n^2\right)b^t(n^2, n^3)\\ &\cdots f_{i_K}^t\left(n_0^k, n^k\right)b^t(n^k, n^1)\end{aligned}$$

In the above expressions $t = 1/k$ and

$$a(n_0, n) = \left(\varphi_{n_0}^0 \otimes \varphi_n, A\varphi_{n_0}^0 \otimes \varphi_n\right)_{H_0\otimes H_+}$$

By taking the limit $k \to \infty$ we get

$$\langle (\theta A) A \rangle \geqslant 0 \quad \blacksquare$$

Remark 2.2. (1) The assumptions of the theorem imply the possibility of having a path space formulation for the Abelian algebra $\hat{\mathfrak{A}}$ (see Ref. 11) which is implicit in the proof through the use of Trotter's formula. Therefore the systems considered are under some aspects "classical" ones.

(2) The possibility of having reflection positivity for subalgebras of quantum systems is mentioned in Refs. 3 and 5.

Remark 2.3. The typical application of the above result for quantum spin systems is the following. Let the Hamiltonian of quantum spin system be of the form

$$H_\Lambda = H_{0,\Lambda} + V_\Lambda \tag{2.11a}$$

where

$$H_{0,\Lambda} = \sum_{R \subset \Lambda} J(R) S_3(R) \tag{2.11b}$$

and

$$V_\Lambda = -a \sum_{x \in \Lambda} S_1(x) \tag{2.11c}$$

In the above $S_i(x)$, $i = 1, 2, 3$, $x \in \Lambda \subset \mathbb{Z}^\nu$ are spin operator

$$\mathbf{S}^2(x) = \sum_{i=1}^{3} S_i(x)^2 = S(S+1)$$

$$\left[S_i(x), S_j(y) \right] = i\epsilon_{ijk} S_k(x) \delta_{x,y}$$

and, for $R \subset \Lambda$

$$S_i(R) = \prod_{x \in R} S_i(x)$$

If $\hat{\mathfrak{A}}$ is the algebra generated by $S_3(R)$, $R \subset \Lambda$ and $\langle \cdot \rangle_{H_0}$ is $\hat{\mathfrak{A}}$-reflection positive with reflections on a plane π_0 containing (or not) sites of the lattice then $\langle \cdot \rangle_H$ is also $\hat{\mathfrak{A}}$-reflection positive with respect to the same reflection operation. (Notice that for planes not containing sites our result would follow from the general theory of reflection positivity as developed in Ref. 4 without having to restrict to the Abelian subalgebra $\hat{\mathfrak{A}}$.) As examples of the above structure we mention antiferromagnets of the type discussed in this paper, the Pirogov–Sinai model[8] with a transverse external field, and the Ising model in triangular lattices with a transverse external field.

Our results also apply to a quantum version of the anharmonic crystal extending the results of Ref. 3 for the classical version. This will be the subject of a subsequent paper.

3. INFRARED BOUNDS

As a typical example requiring the theory of Section 2, we shall treat in this section and the next the Fisher-stabilized Ising antiferromagnet[5] in a magnetic field with both a parallel and a transverse component. In contrast to Ref. 5, the "next-nearest neighbor" interaction is taken for simplicity to be along lattice lines. The Hamiltonian is given by (2.11), with

$$J(R)\begin{cases} +J>0 & \text{if } R=\{x,y\} \text{ nearest neighbors } (|x-y|=1) \\ -\epsilon<0 & \text{if } R=\{x,y\} \text{ next-nearest neighbors along a} \\ & \text{lattice line } (|x-y|=2) \\ -h<0 & \text{if } R=\{x\} \\ \text{zero otherwise} \end{cases} \tag{3.1}$$

The sign of a is irrelevant, and we shall take $a>0$ in (2.11c). Because of the nonzero parallel field h, $\hat{\mathcal{C}}$-generalized reflection positivity (henceforth RP for short) holds only in planes containing sites. The proof in Theorem 2.1 through the Trotter product formula is used here in a twofold way, both through the results of Section 2 and for the purpose of proving certain correlation inequalities. The latter are used to obtain bounds on expectation values of certain operators, which seem difficult to obtain by other means (see Lemma 3.1). In the present model, and in the Pirogov–Sinai model with transverse field mentioned in Remark 2.3, these expectation values are not identically zero, owing to the absence of the symmetry $S_3(x)\to -S_3(x)$. The symmetry-breaking interactions in these models are just those responsible for the lack of RP in planes between sites, so that these features are intimately related.

By Theorem 2.1 we obtain the infrared bound

$$(\hat{S}_3(p)^*, \hat{S}_3(p))_D \leqslant \frac{1}{2\beta\epsilon E_2(p)}, \qquad p\in\Lambda_1^*, \qquad p\neq 0 \tag{3.2}$$

where

$$E_2(p)\equiv\sum_{j=1}^{\nu}(1-\cos 2p_j)$$

$$\hat{S}_3(p)\equiv(2/\Lambda)^{1/2}\sum_{x\in A}e^{-ipx}S_3(x)$$

Above, A denotes the sublattice of the lattice $\Lambda\equiv[-L,L+1]^\nu$ (with periodic boundary conditions, L assumed to be odd) which contains the origin, and

$$\Lambda_1^*=\left\{\frac{\pi}{L+1}n_i,\, n_i=-\frac{L+1}{2}+1,\ldots,\frac{L+1}{2},\, i=1,\ldots,\nu\right\}$$

the Fourier dual of A (viewed as a lattice). We shall also denote by

$$\Lambda^* = \left\{ \frac{\pi}{L+1} n_i, n_i = -L, \ldots, L+1, \quad i = 1, \ldots, \nu \right\}$$

the Fourier dual of Λ, and by B the sublattice complementary to A. We sketch the details leading to (3.2) in Appendix B.

The Duhamel two-point function is defined by

$$(A, B)_D \equiv \frac{1}{\operatorname{Tr} e^{-\beta H}} \int_0^1 dx \operatorname{Tr}\left(e^{x\beta H} A e^{-(1-x)\beta H} B\right)$$

(with $H = H_\Lambda$). We have, on the other hand, the sum rule

$$\frac{2}{\Lambda} \sum_{p \in \Lambda_1^*} \langle \hat{S}_3^*(p) S_3(p) \rangle = 1 \tag{3.3}$$

The connection between (3.2) and (3.3) is realized by the Bruch–Falk inequality[7] (rediscovered in Ref. 2):

$$\frac{(A^*, A)_D}{\frac{1}{2}\langle A^*A + AA^* \rangle} \geqslant f\left[\frac{\beta \langle [A^*, [H, A]] \rangle}{4(\frac{1}{2}\langle A^*A + AA^* \rangle)} \right]$$

where f is the function from $[0, \infty)$ to $[0, 1)$ defined implicitly by

$$f(x \tanh x) = \frac{\tanh x}{x}$$

The function f is monotone decreasing.[2] An easy consequence of the latter property (Theorem 2.2 of Ref. 2) is the fact that if $b \geqslant g f(c/4g)$ with $b, g, c \geqslant 0$ and $b \leqslant b_0$, $c \leqslant c_0$, then $g \leqslant g_0$, where

$$g_0 = \frac{1}{2} (c_0 b_0)^{1/2} \coth\left(\frac{c_0}{4b_0} \right)^{1/2} \tag{3.4}$$

Now,

$$\left[\hat{S}_3^*(p), \left[H_\Lambda, \hat{S}_3(p) \right] \right] = -\frac{4a}{\Lambda} \sum_{x \in \Lambda} S_1(x)$$

and so

$$C \equiv \beta \left\langle \left[\hat{S}_3^*(p), \left[H_\Lambda, \hat{S}_3(p) \right] \right] \right\rangle \leqslant 2a\beta \equiv C_0 \tag{3.5}$$

Therefore, from (3.2), (3.4), (3.5), and the Bruch–Falk inequality we have

$$\langle \hat{S}_3(p)^* S_3(p) \rangle \leqslant \left[\frac{a}{2\epsilon E_2(p)} \right]^{1/2} \coth\left\{ \beta \left[2a\epsilon E_2(p) \right]^{1/2} \right\} \qquad p \in \Lambda_1^*, p \neq 0 \tag{3.6}$$

If we set

$$\rho_0^\Lambda \equiv \frac{2}{\Lambda} \langle \hat{S}_3(0)^* \hat{S}_3(0) \rangle = \frac{2}{\Lambda} \langle \hat{S}_3(0)^2 \rangle$$

we obtain from (3.3) and (3.6) the inequality

$$\rho_0^\Lambda \geqslant 1 - \frac{2}{\Lambda} \sum_{\substack{p \neq 0 \\ p \in \Lambda_1^*}} \left[\frac{a}{2\epsilon E_2(p)} \right]^{1/2} \coth\left\{ \beta\left[2a\epsilon E_2(p)\right]^{1/2} \right\} \tag{3.7}$$

Define, now, the quantities

$$\begin{aligned} \tilde{\rho}_0^\Lambda &= \frac{2}{\Lambda} \left\langle \left[\hat{S}_3(0) - \langle \hat{S}_3(0) \rangle \right]^2 \right\rangle \\ &= \rho_0^\Lambda - \frac{2}{\Lambda} \langle \hat{S}_3(0) \rangle^2 \end{aligned} \tag{3.8}$$

$$g(\beta, h, a) \equiv \left| \frac{h}{(a^2 + h^2)^{1/2}} \tanh\left[\beta(a^2 + h^2)^{1/2} \right] \right|^2 \tag{3.9}$$

$$J(\nu) \equiv \frac{1}{(\pi)^\nu} \int_{B_\nu} d^\nu p \, \frac{1}{2E_2(p)} = \frac{1}{(2\pi)^\nu} \int_{B_\nu^1} d^\nu p \, \frac{1}{2\sum_{j=1}^{\nu}(1 - \cos p_j)} \tag{3.10}$$

where

$$\begin{aligned} B_\nu \equiv [-\pi/2, \pi/2]^\nu, \qquad & B_\nu^1 \equiv [-\pi, \pi]^\nu \\ \tilde{\rho}_0 \equiv \lim_{\Lambda \to \infty} & \tilde{\rho}_0^\Lambda \end{aligned} \tag{3.11}$$

Lemma 3.1.

$$\frac{2\langle \hat{S}_3(0) \rangle^2}{\Lambda} \leqslant g(\beta, h, a)$$

if $\epsilon \leqslant J$.

We shall prove this lemma at the end of this section. Assuming it for the moment, we are ready to prove the main result of this section:

Proposition 3.1. The model defined by (2.11) and (3.1) has a phase transition characterized by

$$\tilde{\rho}_0 > 0 \tag{3.12}$$

in the region of parameters (β, a, h, ϵ) defined by the inequalities

$$\frac{1}{(\pi)^\nu} \int_{B_\nu} d^\nu p \left[\frac{a}{2\epsilon E_2(p)} \right]^{1/2} \coth\left\{ \beta\left[2a\epsilon E_2(p)\right]^{1/2} \right\} < 1 - g(\beta, h, a) \tag{3.13}$$

$$0 < \epsilon \leqslant J \tag{3.14}$$

Proof. It follows from (3.7), (3.8), and Lemma 3.1 [which is true provided (3.14) holds] that

$$\tilde{\rho}_0^\Lambda \geqslant 1 - g(\beta, h, a) - \frac{2}{\Lambda} \sum_{\substack{p \neq 0 \\ p \in \Lambda_1^*}} \left[\frac{a}{2\epsilon E_2(p)} \right]^{1/2} \coth\left\{ \beta\left[2a\epsilon E_2(p)\right]^{1/2} \right\}$$

Taking the limit $\Lambda \to \infty$ in the above inequality, we see that (3.12) will be satisfied if (3.13) is assumed. ■

Remark 3.1. The integral in the left-hand side of (3.13) is finite if $\nu \geqslant 3$ as the inequality[2]

$$\coth x \leqslant 1/x + 1 \tag{3.15}$$

shows.

Remark 3.2. The same estimate (3.13), with $g = 0$, $\epsilon \to J$, $E_2(p) \to E_1(p) = J\sum_{i=1}^{\nu}(1 - \cos p_j)$ may be applied to the Ising model with transverse field and nearest-neighbor interactions of strength J considered in Ref. 6, leading to an improvement of the estimates found there.

Corollary 3.1. Inequality (3.12) is true if $\beta > \tilde{\beta}_c$, where $\beta = \tilde{\beta}_c$ is the unique solution of

$$1 - \alpha - \frac{1}{(\pi)^\nu}\int_{B_\nu} d^\nu p\left[\frac{a}{2\epsilon E_2(p)}\right]^{1/2}\coth\left\{\beta\left[2a\epsilon E_2(p)\right]^{1/2}\right\} = 0 \tag{3.16}$$

provided (3.14) holds and, in addition,

$$g(\beta, h, a) \leqslant \alpha \tag{3.17}$$

and

$$I(a,\epsilon) \equiv \frac{1}{(\pi)^\nu}\int_{B_\nu} d^\nu p\left[\frac{a}{2\epsilon E_2(p)}\right]^{1/2} < 1 \tag{3.18}$$

Above, α is an arbitrary number such that $0 < \alpha < 1$.

Proof. Using inequality (3.15) together with the dominated convergence theorem (for $\nu \geqslant 3$) we see that the left-hand side of (3.16) increases monotonically from $-\infty$ to $[1 - \alpha - I(a,\epsilon)]$ as β varies from 0 to ∞. Hence, there will be a unique solution of (3.16) if and only if (3.18) holds. The final assertion follows then from (3.13) and (3.17). ■

Corollary 3.2. Inequality (3.12) holds if, in addition to (3.14) and (3.17), the following inequality is satisfied:

$$\left(\frac{a}{\epsilon}\right)^{1/2}\left[I(\nu)\right]^{1/2} + \frac{2}{\beta\epsilon}I(\nu) < 1 - \alpha \tag{3.19}$$

Proof. By (3.13), (3.17), and (3.15), (3.12) holds if

$$\frac{1}{\pi^\nu}\int_{B_\nu} d^\nu p\left[\frac{a}{2\epsilon E_2(p)}\right]^{1/2}\left\{\frac{1}{\beta\left[2a\epsilon E_2(p)\right]^{1/2}} + 1\right\} < 1 - \alpha \tag{3.20}$$

By the Schwartz inequality

$$\frac{1}{\pi^\nu}\int_{B_\nu} d^\nu p\left[\frac{a}{2\epsilon E_2(p)}\right]^{1/2} \leqslant \left(\frac{a}{\epsilon}\right)^{1/2}\left[I(\nu)\right]^{1/2} \tag{3.21}$$

and (3.19) follows from (3.20) and (3.21). ■

Remark 3.3. Condition (3.17) is satisfied in particular if

$$a \geqslant \left(\frac{1-\alpha}{\alpha}\right)^{1/2} h$$

independently of β.

Remark 3.4. Proposition 3.1 is of interest because it provides in general better estimates for the region of parameters (β, a, h, ϵ) where a phase transition occurs than those obtained by the Peierls-type arguments outlined in the next section.

We now prove Lemma 3.1. The proof is based on inequalities introduced in Ref. 6 and on the FKG inequality.[9]

Hamiltonian (3.1) may be written

$$\begin{aligned} H_\Lambda = \frac{1}{2} J \sum_{|x-y|=1} S_3(x) S_3(y) - h \sum_{x\in\Lambda} S_3(x) \\ - \frac{\epsilon}{2} \sum_{|x-y|=2} S_3(x) S_3(y) - a \sum_{x\in\Lambda} S_1(x) \end{aligned}$$

Let A be the sublattice containing $\{0\}$ and B its complement with respect to $\mathbb{Z}^\nu$. By a rotation of π around the 1-axis of the spins in $B \cap \Lambda$, H_Λ is transformed to

$$\begin{aligned} H'_\Lambda \equiv -\frac{1}{2} J \sum_{|x-y|=1} S_3(x) S_3(y) - \sum_{x\in\Lambda} h(x) S_3(x) \\ - \frac{\epsilon}{2} \sum_{|x-y|=2} S_3(x) S_3(y) - a \sum_{x\in\Lambda} S_1(x) \end{aligned}$$

where $h(\cdot)$ is an alternating (staggered) magnetic field:

$$h(x) = \begin{cases} h & \text{if } x \in A \\ -h & \text{if } x \in B \end{cases}$$

Let now

$$\begin{aligned} \tilde{H}_\Lambda(\lambda) \equiv H'_\Lambda + \lambda J \sum_{|x|=1} S_3(0) S_3(x) \\ + \lambda\epsilon \sum_{|x|=2} S_3(0) S_3(x) \end{aligned}$$

and $\langle \cdot \rangle'_\lambda$ denote the expectation value in the Gibbs state defined by $\tilde{H}_\Lambda(\lambda)$. In particular, $\langle \cdot \rangle'_{\lambda=0}$ is the Gibbs state defined by H'_Λ and $\langle \cdot \rangle^1_{\lambda=1}$ is the state defined by $\tilde{H}_\Lambda(1)$, where the spin at $x = 0$ is "decoupled" from its neighbors. By Theorem 2.1,

$$\langle S_3(0)\rangle'_\lambda = \lim_{k\to\infty} \langle S(0,1)\rangle_\lambda^{(k)} \tag{3.22}$$

where $\langle S(0,1)\rangle_\lambda^{(k)}$ is the expectation value of the "classical" spin variable $S(0,1)$ corresponding to $S_3(0)$ in the Ising model in $(\nu+1)$ dimensions which results from the proof of Theorem 2.1 for this case [k being the index counting the number of iterations in the Trotter product formula, as in (2.9)]. For explicit formulas, see, e.g., Ref. 6. The only explicit result we shall need concerns the sign of the coupling constant in the $(\nu+1)$-th dimension[6]:

$$\Gamma_k = -\tfrac{1}{2}\log\tanh(\beta a/k) \tag{3.23}$$

where k is the same index above.

Lemma 3.2. (a)

$$\operatorname{sgn}\langle S_3(x)\rangle'_\lambda = \operatorname{sgn} h(x) \qquad \forall\lambda\in[0,1]$$

(b) If $\epsilon \leqslant J$,

$$\langle S_3(x)\rangle'_{\lambda=0} \leqslant \langle S_3(x)\rangle'_{\lambda=1} \tag{3.24a}$$

if $x \in A \cap \Lambda$,

$$\langle S_3(x)\rangle'_{\lambda=0} \geqslant \langle S_3(x)\rangle'_{\lambda=1} \tag{3.24b}$$

if $x \in B \cap \Lambda$. Further

$$\begin{aligned}\langle S_3(x)\rangle'_{\lambda=1} &= \frac{h(x)}{\left[a^2+h(x)^2\right]^{1/2}}\tanh\left\{\beta\left[a^2+h(x)^2\right]^{1/2}\right\}\\ &= \operatorname{sgn} h(x)\times g(\beta,h,a)\end{aligned} \tag{3.25}$$

Proof. Part (a) follows from (3.22) and well-known results for the classical Ising model.

As for part (b) we have, $\forall\lambda, k$

$$\begin{aligned}\frac{d\langle S(0,1)\rangle_\lambda^{(k)}}{d\lambda} &= \beta J\sum_{|x|=1}\Big[\big\langle S(0,1)^2 S(x,1)\big\rangle_\lambda^{(k)}\\ &\qquad - \langle S(0,1)\rangle_\lambda^{(k)}\langle S(0,1)S(x,1)\rangle_\lambda^{(k)}\Big]\\ &\quad + \beta\epsilon\sum_{|x|=2}\Big[\big\langle S(0,1)^2 S(x,1)\big\rangle_\lambda^{(k)} - \langle S(0,1)\rangle_\lambda^{(k)}\\ &\qquad \times\langle S(0,1)S(x,1)\rangle_\lambda^{(k)}\Big]\end{aligned} \tag{3.26}$$

For any $\lambda \in [0,1]$ the N-body interactions in the above Ising model ($N \geqslant 2$) are ferromagnetic for sufficiently large k, because of (3.23). The FKG inequality[9] therefore applies and we obtain

$$\langle S(0,1)S(x,1)\rangle_\lambda^{(k)} \geqslant \langle S(0,1)\rangle_\lambda^{(k)} \langle S(x,1)\rangle_\lambda^{(k)} \tag{3.27}$$

and by part (a)

$$\langle S(0,1)\rangle \geqslant 0 \tag{3.28}$$

Putting (3.27) and (3.28) into (3.26) yields

$$\begin{aligned}\frac{d\langle S(0,1)\rangle_\lambda^{(k)}}{d\lambda} \leqslant{} & \beta J\left\{1-\left[\langle S(0,1)\rangle_\lambda^{(k)}\right]^2\right\} \\ & \times z_\nu \cdot \langle S(x,1)\rangle_{x\in B\cap\Lambda} + \beta\epsilon\left\{1-\left[\langle S(0,1)\rangle_\lambda^{(k)}\right]^2\right\} \\ & \times z_\nu \langle S(x,1)\rangle_{x\in A\cap\Lambda}\end{aligned} \tag{3.29}$$

where z_ν is the number of nearest (and next-nearest) neighbors of a lattice point in ν dimensions. By part (a) and translation invariance [with simultaneous change of sgnh(x)] we have

$$\langle S(x,1)\rangle_{x\in A\cap\Lambda} = -\langle S(x,1)\rangle_{x\in B\cap\Lambda} \equiv S > 0 \tag{3.30}$$

By (3.29) and (3.30) we have

$$\begin{aligned}\frac{d\langle S(0,1)\rangle_\lambda^{(k)}}{d\lambda} &\leqslant -z_\nu\beta(J-\epsilon)\cdot(1-S^2)S \\ &\leqslant 0\end{aligned}$$

if $\epsilon \leqslant J$. This holds for any (sufficiently large) k, hence also in the limit $k \to \infty$. The proof of (3.24b) is identical. ■

Proof of Lemma 3.1.

$$\begin{aligned}\frac{2}{\Lambda}\langle \hat{S}_3(0)\rangle^2 &= \frac{2}{\Lambda}\left[\left(\frac{2}{\Lambda}\right)^{1/2}\left\langle \sum_{x\in A} S_3(x)\right\rangle\right]^2 \\ &= \frac{4}{\Lambda^2}\left[\sum_{x\in A}\langle S_3(x)\rangle_{\lambda=0}^1\right]^2 \\ &< g(\beta,h,a)\end{aligned}$$

by (3.24) and (3.25). ■

Remark 3.5. As remarked in the Introduction, inequalities of the above type were suggested to us by inspection of the similar structure of the spherical model, which we discuss for completeness in Appendix A.

4. THE PEIERLS ARGUMENT

This section is very descriptive, because we only verify the assumptions necessary to apply the general results of Refs. 3 and 5. We consider as typical example the same model (3.1) but with $\epsilon \equiv 0$. Apart from a constant, the Hamiltonian may be written

$$H_\Lambda \equiv (J/2) \sum_{|x-y|=1} S_3(x) S_3(y) - h \sum_{x \in \Lambda} S_3(x) - a \sum_{x \in \Lambda} S_1(x) \quad (4.1)$$

At each site $x \in \mathbb{Z}^{\nu=2}$ let $P^{\pm}(x)$ be the orthogonal projection operators, which project onto the subspaces of states $|\pm)_x$ at x with $S_3(x)|\pm)_x = \pm|\pm)_x$. By combining the methods of Refs. 3 and 5, the following result may be proved:

Proposition 4.1. Let $|h| < J$. Then there exist $0 < \beta_c(J) < \infty$ and $0 < a_0(J) < \infty$ such that, if $\beta > \beta_c(J)$ and $a < a_0(J)$ the following inequality holds:

$$\left\langle P^{\pm}(x) P^{\mp(-1)^{|y|}}(x+y) \right\rangle < 1/4 \quad (4.2)$$

Remark. As in Ref. 5, (4.2) implies the existence of more than one equilibrium state.

Proof. The proof follows from the method of Ref. 3 (as applied to the quantum antiferromagnet) together with the general RP result of Section 2, and the following remarks. As in Ref. 3 we use a contour argument but now draw contours between nearest-neighbor spins if they have the same sign. The relevant "universal projection" P_Λ is of the form

+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+

$(N = 8, M = 4)$

or $(+ \to -)$

(compare Ref. 3, p. 241). Let $e_0(a)$ be the ground-state energy of H_Λ, and define (in close analogy to Ref. 3, p. 256)

$$B_\Lambda \equiv a \sum_{x\in\Lambda} S_1(x)$$

$$H_\Lambda^Z \equiv H_\Lambda - B_\Lambda$$

$$A_\Lambda \equiv H_\Lambda^Z - e_0(a=1)$$

Again here $\pm B_\Lambda \leqslant aA_\Lambda$ by the variational principle. In analogy to Ref. 3, (3.24), p. 256, define

$$\rho \equiv e_0^Z - e_0(a=1) + n\,\Delta\Lambda$$

where $e_0^z = e_0(a=0)$. Then (Ref. 3, p. 257)

$$\epsilon^z(P_\Lambda) \equiv \inf \operatorname{spec}(P_\Lambda H_\Lambda^z P_\Lambda) - e_0(a=1)$$

is the minimal A_Λ energy of any state in $P_\Lambda \mathcal{H}$ and satisfies

$$\epsilon^z(P_\Lambda) - \rho \geqslant (2J - n\Delta)\Lambda \tag{4.3}$$

To prove (4.3), it suffices to recall[5] that, if $|h| < J$, the ground state of H_Λ^z is doubly degenerate, obtained by periodizing the block $\binom{+\,-}{-\,+}$, and also by translating the resulting state by one unit. With the above result, the proof is straightforward along the lines of Ref. 3. ■

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APPENDIX A: THE SPHERICAL MODEL WITH STAGGERED EXTERNAL FIELD–A MOTIVATING EXAMPLE

Some results of Section 3 (Lemma 3.1, for instance) were motivated by the following analogy of the spherical model in the presence of a staggered external field.

In a finite volume $\Lambda \subset \mathbb{Z}^\nu$ we consider a classical "spin" variable $\phi(x) \in \mathbb{R}$ at each site $x \in \Lambda$. For simplicity we take Λ to be the hypercube $\Lambda = \{-L+1, 0, \ldots, L\}^\nu$. The energy $H_\Lambda(\phi)$ of a configuration $\phi: \Lambda \to \mathbb{R}$ is given by

$$H_\Lambda(\phi) = (\phi, [-\Delta/2 - \mu]\phi) - (h, \phi) \tag{A.1}$$

where (a) the "lattice Laplacean" Δ is given by

$$(-\Delta\phi)(x) = 2\nu\phi(x) - \sum_{i=1}^{\nu} [\phi(x+e_i) + \phi(x-e_i)] \tag{A.2}$$

The e_i, $i = 1, \ldots, \nu$, being the unit vectors in the ith direction of $\mathbb{Z}^\nu$, with translations defined by periodicity in Λ.

(b) The scalar product $(\cdot, \cdot)$ is defined by

$$(f, g) \equiv \sum_{x \in \Lambda} \overline{f(x)}\, g(x) \tag{A.3}$$

for any $f, g : \Lambda \to \mathbb{C}$.

(c) $h : \Lambda \to \mathbb{R}$ is the external field

$$h(x) = \begin{cases} +h & \text{if } x \in \Lambda_e, \text{ i.e., } \sum_{i=1}^{\nu} x_i \text{ is even} \\ -h & \text{if } x \in \Lambda_0, \text{ i.e., } \sum_{i=1}^{\nu} x_i \text{ is odd} \end{cases} \tag{A.4}$$

(d) The "chemical potential" $\mu = \mu_\Lambda(\beta, h) < 0$ is introduced in order to handle the spherical constraint, i.e., $\mu_\Lambda(\beta, h)$ solves the equation

$$(1/\Lambda)\langle(\phi, \phi)\rangle_\Lambda = 1 \tag{A.5}$$

where $\langle\ \rangle_\Lambda$ refers to the expectation value in the Gibbs state defined by H_Λ at inverse temperature β.

For $f : \Lambda \to \mathbb{C}$ we define its Fourier transform

$$\hat{f}(p) \equiv \frac{1}{\sqrt{\Lambda}} \sum_{x \in \Lambda} e^{-ipx} f(x) \tag{A.6}$$

for

$$p \in \Lambda^* = \{p = x\pi/L, x \in \Lambda\}.$$

The Hamiltonian H_Λ reads then

$$H_\Lambda(\phi) = \sum_{k \in \Lambda^*} [\omega(k) - \mu]\hat{\phi}(k)^*\hat{\phi}(k) + h\sqrt{\Lambda}\,\hat{\phi}(\pi) \tag{A.7}$$

where

(a)
$$\omega(k) = \sum_{i=1}^{\nu} (1 - \cos k_1) \tag{A.8}$$

and (b) $\hat{\phi}(k)^*$ denotes the complex conjugate of $\hat{\phi}(k)$ and so $\hat{\phi}(k)^* = \hat{\phi}(-k)$.

Since only Gaussian integrations are involved the correlation functions can be obtained explicitly from the two-point functions:

$$\langle\hat{\phi}(k)^*\hat{\phi}(k)\rangle_\Lambda = \frac{1}{2\beta[\omega(k) - \mu]}, \qquad k \neq \pi \tag{A.9}$$

$$\langle\hat{\phi}(\pi)^*\hat{\phi}(\pi)\rangle_\Lambda = \frac{1}{2\beta}\,\frac{1}{\omega(\pi) - \mu} + \frac{h^2\Lambda}{4[\omega(\pi) - \mu]^2}$$

The sum rule (A.5) can then be written as

$$\frac{1}{\Lambda}\langle\hat{\phi}(0)^*\hat{\phi}(0)\rangle_\Lambda = 1 - \frac{h^2}{4[\omega(\pi)-\mu]^2} + \frac{1}{\beta}\frac{1}{\Lambda}\sum_{\substack{k\in\Lambda^* \\ k\neq 0}} \frac{1}{2[\omega(k)-\mu]} \tag{A.10}$$

Therefore in the thermodynamic limit

$$\lim_{\Lambda\to\infty}\frac{1}{\Lambda}\langle\hat{\phi}(0)^*\hat{\phi}(0)\rangle_\Lambda > 0 \tag{A.11}$$

iff

$$\frac{|h|}{2\omega(\pi)} = \frac{|h|}{4\nu J} < 1 \tag{A.12}$$

and

$$\beta > I(\nu) \equiv \frac{1}{(2\pi)^\nu}\int_{B_\nu}\frac{d^\nu p}{2\omega(p)}$$

Since

$$\langle\hat{\phi}(0)\rangle = 0 \tag{A.13}$$

(A.12) implies long-range order. The existence of spontaneous (uniform) magnetization in this case can be obtained explicitly as for instance in Ref. 12 or from the general theorems of Ref. 2.

APPENDIX B

We sketch here the derivation of (3.2). We assume for simplicity that $a = 0$: the modifications introduced by the transverse field are special cases of standard methods.[4]

Our starting point is the generalized Schwartz inequality,[4] which is valid for reflections in planes containing or not containing lattice sites:

$$\left|\left\langle \exp\left(A + \theta B + \sum_{i=1}^{N} C_i\theta D_i\right)\right\rangle_0\right|^2$$
$$\leqslant \left\langle \exp\left(A + \theta A + \sum_{i=1}^{N} C_i\theta C_i\right)\right\rangle_0^2 \left\langle \exp\left(B + \theta B + \sum_{i=1}^{N} D_i\theta D_i\right)\right\rangle_0 \tag{B.1}$$

where

$$A, B, C_i, D_i \in \mathscr{A}_+, \; i = 1, \ldots, N$$

A corollary of (B.1) is

$$\left\langle \exp\left\{ A + \theta B - \sum_{i=1}^{N} \left[C_i + h_i - (\theta C_i + h_i') \right]^2 \right\} \right\rangle_0^2$$
$$\leqslant \left\langle \exp\left[A + \theta A - \sum_{i=1}^{N} (C_i - \theta C_i)^2 \right] \right\rangle_0 \left\langle \exp\left[B + \theta B - \sum_{i=1}^{N} (C_i - \theta C_i)^2 \right] \right\rangle_0 \tag{B.2}$$

For $a = 0$ the system is classical and configurations σ are defined by functions $\sigma : \Lambda \to \mathbb{R}$. The Hamiltonian for one such configuration may be written

$$H_\Lambda(\sigma) = H_{\Lambda,1}(\sigma) + H_{\Lambda,2}(\sigma)$$
$$H_{\Lambda,1}(\sigma) = \tfrac{1}{2} J(\sigma, -\Delta_1 \sigma) - h \sum_{x \in \Lambda} \sigma(x)$$
$$H_{\Lambda,2}(\sigma) = \tfrac{1}{2} \epsilon(\sigma, -\Delta_2 \sigma)$$

where we introduced for convenience the "difference Laplaceans"

$$(-\Delta_n f)(x) = \sum_{i=1}^{\nu} \left[2f(x) - f(x + ne_i) - f(x - ne_i) \right]$$

where e_i, $i = 1, \ldots, \nu$, denote unit vectors along the ν directions in $\mathbb{Z}^\nu$. The partition function Z_Λ is given by

$$Z_\Lambda = \sum_\sigma e^{-\beta H_\Lambda(\sigma)}$$

Let now $Z_\Lambda(h)$ denote the partition function of a system where the variables $\sigma(x)$ in $H_{\Lambda,2}$ are replaced by $\sigma(x) + h(x)$ *for x in some sublattice*, say B. Explicitly,

$$Z_\Lambda(h) = e^{\beta\epsilon(h, -\Delta_2 h)} \sum_\sigma e^{-\beta H_\Lambda(\sigma)} e^{\beta\epsilon(\sigma, \Delta_2 h)} \tag{B.3}$$

where the scalar product is defined by $(f, g) = \sum_{x \in B} f(x) g(x)$ or, alternatively, in the usual way as a sum over the whole lattice, but with the proviso that $h(x) \equiv 0$ if $x \in \Lambda / B$.

Proposition B.1.

$$Z_\Lambda(h) \leqslant Z_\Lambda(0) \tag{B.4}$$

Proof. We shall sketch the proof for $\nu = 1$, the general case being a straightforward extension. The presence of the external field in $H_{\Lambda,1}$ forces us to use reflections in planes containing sites. The operation of reflection about the plane containing the sites 0 and $L+1$ of A (we assume L odd) is defined:

$$(\theta\sigma)(x) = \begin{cases} \sigma(-x) & 0 < |x| \leqslant L \\ \sigma(x) & \text{if } x = 0 \text{ or } x = L+1 \end{cases}$$

We have then

$$H_{\Lambda,1} = A + \theta A$$

$$A = \frac{J}{2} \sum_{x=1}^{L} \left[(\partial_1\sigma)(x)\right]^2 - h \sum_{x=0}^{L+1} \sigma(x)$$

where

$$(\partial_n f)(x) \equiv f(x+n) - f(x), \qquad (\partial_n^* f)(x) \equiv f(x-n) - f(x)$$

$$H_{\Lambda,2}(\sigma + h) = B + \theta D + \tfrac{1}{2}\epsilon(C_1 - \theta C_1 + a_1 - a_1')^2 + \tfrac{1}{2}\epsilon(C_2 - \theta C_2 + a_2 - a_2')^2$$

where

$$B = \frac{\epsilon}{2} \sum_{x \in [0, L-1] \cap B} \left[\partial_2(\sigma - h)(x)\right]^2$$

$$D = \frac{\epsilon}{2} \sum_{x \in [0, L] \cap B} \left[\partial_2^*(\sigma - h)(\theta x)\right]^2$$

and where

$$C_1 = \sigma(1), \qquad C_2 = \sigma(L)$$

$$a_1 = h(1), \qquad a_2 = h(L)$$

$$a_1' = h(-1), \qquad a_2' = h(-L)$$

Using (B.2) we obtain

$$Z_\Lambda(h) \leqslant Z_\Lambda(h_+)^{1/2} Z_\Lambda(h_-)^{1/2}$$

where

$$h_+(x) \equiv \begin{cases} h(x), & x \in [0, L+1] \cap B \\ h(-x), & x \in [-L, -1] \cap B \end{cases}$$

$$h_-(x) \equiv \begin{cases} h(x), & x \in [-(L+1), 0] \cap B \\ h(-x), & x \in [1, L] \cap B \end{cases}$$

Repeating the procedure, considering reflections in the planes containing successively the sites $\{2, -L+1\}$, $\{4, -L+3\}, \ldots$ of A, we finally obtain (B.4). ■

Remark B.1. One might ask why reflections in planes containing sites of B are not permitted (together with the previous reflections about planes containing points of A). The point is that, for instance, a reflection through the plane containing 0 and $L+1$ "eliminates" the quantities $h(1)$, $h(-1)$, $h(L)$, and $h(-L)$. A further reflection in the plane through $\{1, -L\}$ will have as a consequence that a subsequent reflection in the plane containing $\{2, -L+1\}$ will bring back quantities such as $h(1)$ and $h(-L)$, which should already have been disposed of. This does not happen if only reflections through planes containing points of A are considered. This fact is related to the modifications of the chessboard estimates when reflections in planes containing sites are considered (Theorem 4.3 of Ref. 4).

The same proof above yields (B.4) with $h(x) \equiv 0$ for $x \in \Lambda / A$. We use this to obtain (3.2): (B.3) and (B.4) yield

$$\langle e^{\beta\epsilon(\sigma, \Delta_2 h)} \rangle \leqslant e^{\beta\epsilon(h, \Delta_2 h)/2}$$

Putting $h \to \lambda h$ above we find

$$\begin{aligned} \frac{d}{d\lambda} \langle e^{\lambda\beta\epsilon(\sigma,\Delta_2 h)} \rangle \Big|_{\lambda=0} &= \beta\epsilon \langle (\sigma, \Delta_2 h) \rangle = \beta\epsilon \sum_{x\in A} (\Delta_2 h)(x) \langle \sigma(x) \rangle \\ &= \beta\epsilon \langle \sigma(0) \rangle \sum_{x \in A} (\Delta_2 h)(x) = 0 \end{aligned}$$

because of translation invariance and the fact that periodic boundary conditions imply

$$\sum_{x\in A} (\Delta_2 h)(x) = \sum_{x \in B} (\Delta_2 h)(x) = 0$$

Hence

$$\frac{d^2}{d\lambda^2} \langle e^{\lambda\beta\epsilon(\sigma,\Delta_2 h)} \rangle \Big|_{\lambda=0} \leqslant \frac{d^2}{d\lambda^2} e^{\lambda^2 \beta\epsilon(h,\Delta_2 h)/2} \Big|_{\lambda=0}$$

or

$$\beta^2\epsilon^2 \langle (\sigma, \Delta_2 h)^2 \rangle \leqslant \beta\epsilon(h, \Delta_2 h) \tag{B.5}$$

As written, (B.5) holds only for h real, but it extends immediately to complex h:

$$\beta^2\epsilon^2 \langle |(\sigma, \Delta_2 h)|^2 \rangle \leqslant \beta\epsilon |(h, \Delta_2 h)| \tag{B.6}$$

(the above scalar product being antilinear in the first argument). Putting

$$h(x) \equiv \begin{cases} e^{ip\cdot x}, & x \in A \\ 0 & \text{otherwise} \end{cases}$$

We obtain from (B.6) $[\sigma_p \equiv (2/\Lambda)^{1/2}\sum_{x\in A} e^{-ip\cdot x}\sigma(x)]$:

$$\langle \sigma_p \sigma_{-p} \rangle \leqslant \frac{1}{2\beta\epsilon E_2(p)}, \qquad p \in \Lambda_1^*, p \neq 0 \tag{B.7}$$

The above formula corresponds to (3.2) for the classical model. ■

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